

Arithmetic Positivity Structures

Part I of *Arithmetic Positivity Program*

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Abstract

A geometric explanation of classical Weil positivity requires an independently constructed positive structure and an exact comparison with the arithmetic form. We fix the full complex algebra of smooth compactly supported logarithmic test functions and its centered Mellin transform, and express the problem through maps preserving an explicitly normalized Hermitian form. Two sufficient constructions emerge: a realization in a primitive intersection space with an independently established Hodge index inequality, and a polarized cohomological representation whose integrated action has an ordinary positive trace comparison. We prove the algebraic transfer and normalization statements, explain the analytic conditions needed for completion and integration, and distinguish these constructions from equivalent positivity certificates. Finite-field purity, arithmetic Hodge index, cyclic cohomology, and condensed six-functor formalisms provide different parts of the proposed network; none of the inspected sources supplies all its arithmetic comparison maps. An audit of five associated formal research repositories identifies direct logical equivalences, schematic purity predicates, and unrestricted carrier maps that do not establish the required preservation equations. The resulting program specifies the geometric realization, primitive projection, adjoint compatibility, and exact local-to-global comparison as separate mathematical tasks. It proves no new global positivity theorem for the Riemann zeta function and asserts no universally minimal arithmetic geometry.

1 The realization problem

The classical Weil form has several descriptions: a sum over zeros, an arithmetic distribution involving prime powers, and, in suitable constructions, a cohomological trace. These descriptions do not have the same relation to positivity. A convergent sum is a defined number. A trace formula is an equality. A positive Hermitian realization is an additional structure that can make the sign of that number follow from geometry. The distinction determines the problem considered here.

We seek a network of explicit morphisms leading from an independently specified geometric object to the exact classical Weil form. Every arrow must have a source, a target, and a preservation equation. A statement that two propositions are equivalent does not by itself give an arrow between the spaces, operators, or pairings mentioned in those propositions. This remains true when the propositions are motivated by cohomology or by an internal language.

The algebraic endpoint of the program is elementary. If a Hermitian form is the pullback of a nonnegative form, it is nonnegative. Its geometric use is difficult because the pullback identity has to be proved for the correct test functions, with the correct involution and normalization. A

proposed construction must account for the entire explicit formula. In particular, the pole and archimedean terms cannot be removed without an explicit restriction or correction.

The classical Weil criterion identifies all-test positivity with the Riemann hypothesis. We use that established equivalence only to identify the strength of the target and to detect circular hypotheses. We do not use the hypothesis, its Nyman formulation, or an equivalent positivity assertion as an input. Connes and Consani give the relevant normalization and a compact-test formulation of the criterion in their appendices [1]. Thus a global norm-square identity for the exact form would be a positivity certificate of the full target. A useful geometric proposal must explain how its norm and comparison map could be constructed independently.

1.1 Relative sufficiency

The word *minimal* requires a category. The categories of arithmetic schemes, metrized line bundles, Hilbert representations, and cyclic modules have different objects and different morphisms. A terminal object in one of them would say nothing about a purported smallest object in another. Even within a fixed category, existence, generation by test functions, and preservation of a pairing are distinct conditions.

We therefore ask a narrower question. Once a class of geometric objects and its allowed maps has been fixed, which parts of the structure are used by a direct positivity-transfer argument? An answer can be economical without being a theorem of universal minimality. For example, algebraic pullback positivity needs neither completeness nor continuity. A construction by analytic limits generally needs both a topology and an appropriate continuity theorem. Neither condition should be inserted into the algebraic lemma merely because a later application might require it.

There are two principal routes in this paper. The intersection route uses a primitive subspace on which a geometric intersection form is nonpositive. The cohomological route uses a positive pairing and a centered unitary action, followed by a trace comparison for its integrated operators. They share a need for an exact realization of the test algebra, but their construction problems differ. The first asks for arithmetic correspondence classes and an intersection identity. The second asks for an arithmetic cohomology, a polarization, and a trace-ideal comparison.

1.2 Scope of the results

Our results are conditional transfer theorems, explicit algebraic calculations, and an audit of the assumptions required by proposed geometric bridges. The statements of the classical analytic form are mathematical definitions in this paper. The companion Lean files formalize specified algebraic and operator-domain consequences; they do not formalize the full analytic explicit formula or construct an arithmetic realization of it. This distinction is recorded in section 11.

Part II, *Finite Weil Approximations*, studies finite approximations, support restrictions, and the convergence needed to pass quadratic inequalities to the full test space. Part III, *Structural Arithmetic Geometry*, treats the category of positivity certificates and the residual comparison maps. The present part fixes the common classical form and examines geometric sources of its sign. None of the three parts is a proof of global Weil positivity.

2 The centered Weil form

2.1 The test algebra

Definition 2.1. Let

$$\mathcal{T} = C_c^\infty(\mathbb{R}, \mathbb{C}).$$

For $f, g \in \mathcal{T}$, define additive convolution and involution by

$$(f * g)(t) = \int_{\mathbb{R}} f(u)g(t - u) du, \quad f^*(t) = \overline{f(-t)}.$$

The centered Mellin transform, written in logarithmic coordinates, is

$$F_f(z) = \int_{\mathbb{R}} f(t)e^{zt} dt, \quad z \in \mathbb{C}. \quad (1)$$

The topology on $\mathcal{T}$ is the usual locally convex inductive limit, over compact sets $K \subset \mathbb{R}$, of the Fréchet spaces of smooth functions supported in K .

The adjective centered concerns the later evaluation at $z = \rho - 1/2$. Formula (1) is the bilateral Laplace transform in the variable t . We retain Mellin terminology because the substitution $x = e^t$ identifies it with the multiplicative transform used in explicit formulas. The full complex test space is essential: passing to its underlying real vector space preserves every test, whereas restricting to real-valued functions changes the domain.

Lemma 2.2. *For $f, g \in \mathcal{T}$, the function F_f is entire and*

$$F_{f*g}(z) = F_f(z)F_g(z), \quad F_{f^*}(z) = \overline{F_f(-\bar{z})}. \quad (2)$$

The space $\mathcal{T}$ is a commutative complex $*$ -algebra under these operations. If $\text{supp } f \subset [-a, a]$, then $\text{supp}(f * f^*) \subset [-2a, 2a]$.

Proof. On a compact support the exponential and each of its z -derivatives are bounded uniformly on compact subsets of $\mathbb{C}$, which permits differentiation under the integral. Absolute integrability of the double integral proves the convolution formula by the substitution $t = u + v$. In the involution formula put $u = -t$ and conjugate the integral. Associativity follows from the same compact-support Fubini argument, and commutativity follows by changing u to $t - u$. The support of a convolution lies in the sum of the two supports. Reflection preserves $[-a, a]$. $\square$

2.2 Convergence and symmetry

Let $\mathcal{Z}$ be the multiset of nontrivial zeros of the Riemann zeta function, with analytic multiplicities, and write $z_\rho = \rho - 1/2$. The unconditional critical-strip bound places $\Re z_\rho$ in $[-1/2, 1/2]$. The usual zero-counting estimate is $N(T) = O(T \log(T + 2))$. These facts are independent of the assertion that all z_ρ are imaginary.

Lemma 2.3. *Fix $a > 0$ and a nonnegative integer m . There is a continuous seminorm $p_{a,m}$ on the smooth functions supported in $[-a, a]$ such that*

$$|F_f(\sigma + i\tau)| \leq p_{a,m}(f)(1 + |\tau|)^{-m} \quad \text{for } |\sigma| \leq \frac{1}{2}.$$

Consequently $\sum_{\rho \in \mathcal{Z}} |F_f(z_\rho)|$ converges, and its sum is bounded by a continuous seminorm on each fixed-support test space.

Proof. For $|\tau| \geq 1$, integrate m times by parts against $e^{i\tau t}$. The boundary terms vanish because $e^{\sigma t} f(t)$ is smooth and compactly supported. Its m th derivative is $\sum_{j=0}^m \binom{m}{j} e^{\sigma t} \sigma^{m-j} f^{(j)}(t)$. The L^1 norms are bounded uniformly for $|\sigma| \leq 1/2$ by a constant depending on a, m times $\max_{j \leq m} \|f^{(j)}\|_\infty$. The case $|\tau| < 1$ follows from the direct integral bound.

Divide the zeros of height at least one into shells $2^k \leq |\Im \rho| < 2^{k+1}$. The number in the k th shell is at most $C2^k(k+1)$, while the transform is bounded by $p_{a,m}(f)2^{-km}$. Choose $m > 1$ and sum $Cp_{a,m}(f)(k+1)2^{-(m-1)k}$. The finitely many zeros below height one cause no difficulty. The same estimates prove the seminorm bound. $\square$

Definition 2.4. The centered Weil distribution, Hermitian form, and quadratic form are

$$W(h) = \sum_{\rho \in \mathcal{Z}} F_h(z_\rho), \quad (3)$$

$$B_W(f, g) = W(f * g^*), \quad (4)$$

$$Q_W(f) = B_W(f, f). \quad (5)$$

The sum includes multiplicities. The form is linear in its first argument and conjugate-linear in its second.

Absolute convergence justifies reindexing by the zero symmetry $\rho \mapsto 1 - \bar{\rho}$. Since $z_{1-\bar{\rho}} = -\bar{z}_\rho$, equations (2) give $W(h^*) = \overline{W(h)}$. Commutativity of convolution then proves that B_W is Hermitian. In particular, Q_W is real-valued. Its continuity on the test algebra follows from the distribution estimate and continuity of convolution and involution.

The expanded quadratic form is

$$Q_W(f) = \sum_{\rho \in \mathcal{Z}} F_f(\rho - \tfrac{1}{2}) \overline{F_f(\tfrac{1}{2} - \bar{\rho})}. \quad (6)$$

An individual summand in (6) is an absolute square when $\Re \rho = 1/2$. The second factor cannot be replaced by $\overline{F_f(\rho - 1/2)}$ without that condition. The convergence proof deliberately avoids such a replacement.

Remark 2.5. Truncation by $|\Im \rho| \leq T$ preserves the relevant zero symmetries and therefore gives a Hermitian finite form. This is a useful approximation property. Hermitian does not mean nonnegative. Nor does the tail estimate prove that a finite form has the required sign.

2.3 The arithmetic comparison

For $h \in \mathcal{T}$, the explicit formula in the present convention is

$$\begin{aligned} W(h) = & F_h(\tfrac{1}{2}) + F_h(-\tfrac{1}{2}) - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (h(\log n) + h(-\log n)) \\ & - (\log(4\pi) + \gamma)h(0) \\ & - \int_0^\infty \frac{e^{-u/2}(h(u) + h(-u)) - 2e^{-u}h(0)}{1 - e^{-2u}} du. \end{aligned} \quad (7)$$

Here Λ is the von Mangoldt function and γ is Euler's constant. The translation from the multiplicative convention is $g(x) = x^{-1/2}h(\log x)$, so that its Mellin transform is $F_h(s - 1/2)$. Equation (7) is the centered version of the formulas in [1, Appendix B, equations 147-151].

The numerator in the integral vanishes to first order at $u = 0$. Indeed, $h(u) + h(-u) = 2h(0) + O(u^2)$, and the quotient has limit $h(0)/2$. At infinity the compact-support part vanishes

and the counterterm decays exponentially. Thus the displayed integral is an ordinary convergent integral with its value at zero filled in by continuity. The subtraction is part of the functional, not an optional normalization.

For a test f supported in $[-a, a]$, only prime powers $n \leq e^{2a}$ occur after putting $h = f * f^*$. This is exact support localization. It is different from discarding primes whose contribution is small: the other terms are zero because they lie outside the support. The resulting finite arithmetic expression still contains signed terms. Its finiteness supplies no general inequality.

The two pole evaluations on a square are

$$F_{f*f^*}(\tfrac{1}{2}) + F_{f*f^*}(-\tfrac{1}{2}) = 2\Re\left(F_f(\tfrac{1}{2})\overline{F_f(-\tfrac{1}{2})}\right). \quad (8)$$

They vanish on the moment-restricted subspace

$$\mathcal{T}_0 = \{f \in \mathcal{T} : F_f(1/2) = F_f(-1/2) = 0\}.$$

This is a natural analogue of removing trivial cohomological degrees. It is not dense in the ordinary test topology: the evaluations are continuous, so $\mathcal{T}_0$ is closed, and a nonnegative nonzero bump has nonzero evaluation at $1/2$. Passing from positivity on $\mathcal{T}_0$ to the full criterion therefore requires an additional theorem. It cannot be justified by an unstated density argument.

3 Primitive intersection spaces

3.1 Signed pullback

Definition 3.1. A primitive intersection realization of a Hermitian form B on a complex vector space V consists of a complex vector space E , a Hermitian form I on E , a specified subspace $P \subset E$, and a complex-linear map $\Gamma : V \rightarrow P$ satisfying

$$B(f, g) = -I(\Gamma f, \Gamma g). \quad (9)$$

Its index condition is $I(v, v) \leq 0$ for all $v \in P$.

The word intersection in this abstract definition records the intended application; it imposes no geometric origin by itself. For an arithmetic construction, E , I , and P must be specified as actual geometric classes, an intersection pairing, and a primitive condition. Merely renaming $-B$ as an intersection form would satisfy the equality but would not establish the index condition independently.

Theorem 3.2 (Primitive transfer). *A primitive intersection realization satisfying the index condition has $B(f, f) \geq 0$ for every $f \in V$.*

Proof. For each f , the vector Γf belongs to P , so $I(\Gamma f, \Gamma f) \leq 0$. Negating this inequality and using (9) proves the assertion. $\square$

Only image membership and a diagonal comparison are used in this proof. Bilinearity and the off-diagonal comparison make the realization compatible with a larger network of morphisms, but are not needed for the final inequality. Injectivity would be relevant to strict positivity, and completeness to a Hilbert realization. Neither is required for nonnegativity.

There is also an order-theoretic variant. Suppose

$$B(f, f) = -I(\Gamma f, \Gamma f) + R(f)$$

with $R(f) \geq 0$. The same proof gives positivity. This variant can be useful when a geometric model supplies a lower bound with an explicitly positive remainder. An error term whose absolute value is merely bounded does not suffice unless a positive margin dominates it. Part II develops the needed quantitative distinctions.

3.2 Degree normalization

The primitive condition on a product of curves is usually expressed by vanishing degree and codegree. The following calculation separates those two corrections from the index theorem. It is stated over $\mathbb{R}$ to keep the intersection convention bilinear. Complexification then gives the associated Hermitian statement.

Lemma 3.3 (Hyperbolic normalization). *Let I be a symmetric bilinear form on a real vector space E . Suppose $e, f \in E$ satisfy*

$$I(e, e) = I(f, f) = 0, \quad I(e, f) = 1.$$

Set $a(v) = I(v, e)$, $b(v) = I(v, f)$, and

$$\pi v = v - b(v)e - a(v)f. \tag{10}$$

Then $I(\pi v, e) = I(\pi v, f) = 0$, $\pi^2 = \pi$, and

$$I(\pi v, \pi w) = I(v, w) - a(v)b(w) - b(v)a(w). \tag{11}$$

Proof. Pair (10) with e . The result is $a(v) - b(v)I(e, e) - a(v)I(f, e) = 0$. Pairing with f is identical. Thus $a(\pi v) = b(\pi v) = 0$, which gives idempotence. Expand $I(\pi v, \pi w)$ by bilinearity. The two mixed subtractions each occur twice, and the terms from pairing the correction vectors add back one copy of each. The remaining expression is (11). $\square$

Suppose a raw correspondence map $C : V \rightarrow E$ is available. If $\Gamma = \pi C$, the desired equality becomes

$$B(v, w) = -I(Cv, Cw) + a(Cv)b(Cw) + b(Cv)a(Cw). \tag{12}$$

This formulation has practical value: it says exactly which degree terms must appear in a raw geometric comparison. One may prove the normalization identity without constructing C or proving any arithmetic trace formula. The remaining comparison is then a smaller, explicit equality.

For a real symmetric I , its Hermitian extension to $E \otimes_{\mathbb{R}} \mathbb{C}$ is

$$\begin{aligned} I_{\mathbb{C}}(x + iy, u + iv) &= I(x, u) + I(y, v) \\ &\quad + i(I(y, u) - I(x, v)). \end{aligned}$$

In particular, $I_{\mathbb{C}}(x + iy, x + iy) = I(x, x) + I(y, y)$. Hence negativity on a real primitive subspace implies negativity on its complexification. This extension is appropriate when the classical complex tests are realized as pairs of real geometric classes. An assertion for real-valued tests alone would not justify the same conclusion without specifying this extension and its comparison law.

3.3 A nontrivial algebraic example

Take $E = \mathbb{R}^2$ with $I((a, b), (c, d)) = ac - bd$, and let P be the second coordinate axis. The form is negative definite on P . For $V = \mathbb{R}$, define $\Gamma(x) = (0, x)$. Then

$$-I(\Gamma x, \Gamma y) = xy.$$

This realizes the ordinary positive form on $\mathbb{R}$ as a negative primitive intersection. The ambient form is indefinite and the primitive part is nonzero. The example checks that the package does not accidentally require the whole ambient intersection form to be nonpositive or collapse the geometry to a singleton.

The example has no arithmetic comparison with B_W . Its role is to test the formal distinction among ambient signature, primitive membership, and pullback. These distinctions are precisely what a proposed arithmetic instantiation must preserve.

4 The finite-field model

4.1 A genuine surface

Let C be a smooth projective geometrically connected curve over a finite field. Over an algebraic closure, the product $S = C \times C$ is a smooth projective surface. The two fiber classes $e = [P \times C]$ and $f = [C \times P]$ satisfy the hypotheses of theorem 3.3. Their sum is an ample class: it is the external tensor product of positive-degree line bundles on the two factors. A divisor class orthogonal to both fibers is therefore orthogonal to an ample class, and the surface Hodge index theorem gives a nonpositive self-intersection.

This is a geometric realization of the abstract primitive space. It includes an actual dimension-two object, intersection of divisors, and an ample direction. None of these pieces is supplied simply by the fact that a curve has a zeta function. The product over the finite field is part of the construction.

Correspondences add the morphisms needed for arithmetic. Their composition is defined by pulling back to C^3 , intersecting, and pushing forward to the first and third factors. Transposition exchanges the factors. The degree and codegree arise by pairing with the fibers. These operations provide candidates for multiplication, involution, and trivial-degree corrections. The classical trace of a correspondence is its degree plus codegree minus its intersection with the diagonal; the cohomological interpretation isolates the action on H^1 [2, Section 2].

The useful lesson is the list of preservation laws, rather than a claim that every construction resembling this dictionary is automatically valid. A number-field analogue would have to define the counterpart of composition and prove that the test convolution maps to it. It would have to identify the counterpart of transposition with the test involution. Finally it would have to compare the signed primitive intersection with the actual arithmetic distribution.

4.2 The diagonal calculation

For the diagonal class Δ one has $I(\Delta, e) = I(\Delta, f) = 1$. Formula (10) gives

$$\pi\Delta = \Delta - e - f, \quad I(\pi\Delta, \pi\Delta) = I(\Delta, \Delta) - 2.$$

The normal bundle of the diagonal is the tangent bundle of C , whose degree is $2 - 2g$. Therefore

$$I(\pi\Delta, \pi\Delta) = -2g.$$

The negative primitive pairing thus detects the expected middle-degree size. For genus zero the primitive diagonal class is numerically zero; for positive genus the self-intersection is strictly negative. This small calculation explains why removing the two fiber directions is structurally meaningful.

The calculation also fixes a sign. Taking the raw diagonal self-intersection as the positive quantity would give $2 - 2g$, which has neither the right sign in all genera nor the right middle-degree interpretation. The correction by two trivial directions changes it to $-2g$, and negation produces a nonnegative number. The pole corrections in an arithmetic trace formula deserve comparable care.

4.3 Purity and realization

Deligne's theorem gives the expected weights for the cohomology of smooth proper varieties over finite fields [3]. For more general sheaves and nonproper spaces, the statements involve mixed weights and bounds; support and properness hypotheses matter [4]. These are established theorems about specified geometric cohomology groups and specified Frobenius actions.

The application to C is not a template that can be applied to an unnamed global arithmetic object. The finite-field proof already has a Frobenius endomorphism, a finite-dimensional cohomology theory, and compatibility with the zeta function of that variety. In a number-field program, each of those ingredients must be replaced by a construction with its own domain of definition. A family of local Frobenius actions does not alone define a single global real flow or identify its integrated trace with W .

Hesselholt provides a separate, genuine finite-field example: periodic topological cyclic homology gives a regularized-determinant interpretation of zeta functions for smooth proper schemes over finite fields [5]. This shows that higher and topological constructions can realize the desired kind of arithmetic data in a setting where the hypotheses are established. It does not supply the missing global number-field comparison. A determinant realization and a positivity mechanism remain distinct questions even when both are available in a particular finite-field example.

5 Arithmetic intersection theory

5.1 The dimension requirement

The ordinary scheme $\mathrm{Spec} \mathbb{Z}$ has Krull dimension one. Its self-product over itself is again $\mathrm{Spec} \mathbb{Z}$, since $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$. The product in the category of schemes also exists and is the same fiber product: $\mathrm{Spec} \mathbb{Z}$ is terminal in that category. Neither ordinary scheme product produces the surface used in the preceding section. A product over a proposed different base, such as an object over $\mathbb{F}_1$, requires a new category and construction with the stated geometric properties.

An arithmetic surface in the usual sense is a regular projective flat scheme of relative dimension one over $\mathrm{Spec} \mathcal{O}_K$, for a number field K . It has absolute dimension two. Its arithmetic divisors include archimedean metric or Green-function information, and its arithmetic intersection pairing reflects that information. Such surfaces certainly exist, but no particular one is supplied merely by the notation $\mathrm{Spec} \mathbb{Z}$ or by the Euler product of ζ .

One could choose an auxiliary arithmetic surface and ask whether the Weil form is represented by its arithmetic divisor classes. That is a legitimate research problem. The choice would have to be accompanied by a construction of the class assigned to each test and a proof that the resulting intersection equals (7). The existence of a large supply of surfaces is not an existence theorem for that comparison.

5.2 The available index theorem

Arithmetic Hodge index for codimension-one cycles is established under geometric and positivity hypotheses; Moriwaki's work extends the arithmetic-surface setting [6]. An adelic version due to Yuan and Zhang has the following useful inequality [7, Theorem 1.3]. Let X be a normal geometrically connected projective variety of dimension $n \geq 1$ over a number field. Let $\overline{M}$ be an integrable adelic line bundle, and let $\overline{L}_1, \dots, \overline{L}_{n-1}$ be nef adelic line bundles with big underlying line bundles. If

$$M \cdot L_1 \cdots L_{n-1} = 0$$

on the generic variety, then

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0. \quad (13)$$

Their equality statement needs additional positivity and boundedness assumptions. We use only the displayed inequality.

This theorem can supply the index component of theorem 3.1. It does not identify the realization map. At minimum, a proposed application must give the variety X , the fixed nef classes, the integrable class associated with a test, and the generic primitive condition. It must also specify how complex test functions are represented in a real arithmetic intersection space.

The coefficient issue is substantive. The original arithmetic objects are line bundles or rational combinations of classes. A complex-linear map from $\mathcal{T}$ does not literally take values in the abelian group of line bundles. One must pass to an appropriate real vector space and then complexify, or use a completion of classes whose intersection pairing extends continuously. Negativity on rational classes passes to a real completion only after the topology, density, and continuity have been established. These are separate lemmas, not changes of notation.

5.3 A precise arithmetic candidate

Fix an arithmetic intersection space E obtained from such classes, together with a continuous extension I of the arithmetic pairing and a primitive subspace P . Consider a candidate map

$$\Gamma_a : \mathcal{T}_a \longrightarrow P_{\mathbb{C}}, \quad \mathcal{T}_a = \{f \in \mathcal{T} : \text{supp } f \subset [-a, a]\}.$$

An application requires the following identities on overlapping supports:

$$\Gamma_b|_{\mathcal{T}_a} = \Gamma_a \quad (a \leq b), \quad (14)$$

$$B_W(f, g) = -I_{\mathbb{C}}(\Gamma_a f, \Gamma_a g) \quad (f, g \in \mathcal{T}_a). \quad (15)$$

If these maps exist and the index inequality holds on $P_{\mathbb{C}}$, positivity follows for every test, since every compactly supported function belongs to some $\mathcal{T}_a$. No global completeness assumption is needed for that last step. Completeness may have been needed earlier to construct the classes or define their pairings.

For a candidate defined by local arithmetic data, (15) can be divided into the pole evaluations, the finitely many relevant prime powers, and the archimedean integral. This decomposition makes the comparison reviewable. It does not imply that the pieces are separately nonnegative, nor that matching them one by one can be done independently of their common normalization.

An independently proved inequality stronger than (13) could also be useful. For example, a lower bound on $-I_{\mathbb{C}}(\Gamma_a f, \Gamma_a f)$ could absorb a controlled comparison error. Such a bound would need to be uniform over a stated set of tests and compatible with changing a . A pointwise numerical check of selected classes cannot replace this estimate.

6 Polarized cohomological representations

6.1 The geometric pairing

Deninger's cohomological program makes the structural requirements particularly visible. A cup product, a trace in top degree, and a Hodge operator are used to obtain a positive pairing; compatibility with the flow yields a weight identity [8, Section 3]. In the relevant number-field setting these data belong to a conjectural arithmetic formalism. We use that proposal to formulate a conditional construction, not to assume the missing polarization.

Here is a precise algebraic version. Let E be a complex vector space, let L be a one-dimensional complex vector space, let $c : E \times E \rightarrow L$ be bilinear, and let $s : E \rightarrow E$ be conjugate-linear. Let $\tau : L \rightarrow \mathbb{C}$ be linear. Define

$$\langle x, y \rangle = \tau(c(x, sy)). \quad (16)$$

Hermitian symmetry and positive definiteness of (16) are explicit conditions. They do not follow merely from the existence of c, s, τ .

Suppose $V_t : E \rightarrow E$ is a group of complex-linear automorphisms, with an action $A_t : L \rightarrow L$ satisfying

$$c(V_t x, V_t y) = A_t c(x, y), \quad (17)$$

$$s V_t = V_t s, \quad (18)$$

$$\tau(A_t \ell) = e^t \tau(\ell). \quad (19)$$

These equations describe the preservation needed from a cohomological flow. The weight is in the trace on top degree, while the Hodge operator commutes with the flow on the middle degree.

Proposition 6.1 (Conformal pairing). *Under (17)-(19), the pairing satisfies*

$$\langle V_t x, V_t y \rangle = e^t \langle x, y \rangle. \quad (20)$$

If the pairing is positive definite, $U_t = e^{-t/2} V_t$ is an isometric group on the resulting pre-Hilbert space.

Proof. Use $s V_t y = V_t s y$, apply (17) to $x, s y$, and then apply (19). Multiplying both arguments by $e^{-t/2}$ multiplies their pairing by e^{-t} , which cancels the factor in (20). The group law is preserved by the scalar factor, and U_{-t} is the inverse. $\square$

This proposition explains a direct preservation path. A geometric positivity theorem for (16), together with three compatibility identities, produces an isometric centered action. The statement does not explain how to construct those arithmetic data; that is the remaining geometric problem.

6.2 Completion and domains

Lemma 6.2. *Let E be a pre-Hilbert space and U_t a group of linear isometries. Suppose $t \mapsto U_t x$ is norm-continuous for every $x \in E$. Then each U_t extends uniquely to a unitary operator on the Hilbert completion H , and these extensions form a strongly continuous unitary group.*

Proof. An isometry sends Cauchy sequences to Cauchy sequences and equivalent sequences to equivalent sequences. It therefore extends uniquely to an isometry of H . The inverse extends from U_{-t} , proving surjectivity. The group law holds on the dense subspace E , hence on H .

For continuity at a fixed t_0 and a vector $x \in H$, choose $y \in E$ with $\|x - y\| < \varepsilon$. Isometry gives

$$\|U_t x - U_{t_0} x\| \leq 2\varepsilon + \|U_t y - U_{t_0} y\|.$$

The last term tends to zero by hypothesis. Since ε was arbitrary, the extension is strongly continuous. $\square$

At the infinitesimal level, assume a linear operator Θ has domain $D \subset H$ and satisfies

$$\langle \Theta x, y \rangle + \langle x, \Theta y \rangle = \langle x, y \rangle \quad (x, y \in D). \quad (21)$$

Then $A = \Theta - \frac{1}{2}$ is skew-symmetric on D . A statement about skew symmetry is a statement about this domain. It does not assert that A is closed, maximal skew-adjoint, or the generator of a unitary group. Those conclusions require further hypotheses or the completed-group construction above.

Proposition 6.3 (Eigenvector consequence). *Suppose (21) holds and the Hilbert pairing is positive definite. If $0 \neq x \in D$ and $\Theta x = \lambda x$, then $\Re \lambda = 1/2$.*

Proof. Set $y = x$ in (21). Linearity in the first variable and conjugate-linearity in the second give

$$(\lambda + \bar{\lambda})\langle x, x \rangle = \langle x, x \rangle.$$

Since $x \neq 0$, its squared norm is positive. Division gives $2\Re \lambda = 1$. $\square$

The proposition applies to an actual eigenvector in the domain. It neither asserts that the operator has an eigenbasis nor identifies its eigenvalues with zeta zeros. An arithmetic application needs a comparison map carrying the relevant spectral data into this domain with the correct eigenvalue equation. Without that comparison, the conclusion remains an operator-theoretic lemma.

6.3 Integrated operators

Let U be a strongly continuous unitary group on a complex Hilbert space H . For $f \in \mathcal{T}$, define the bounded operator T_f by vector integration,

$$T_f x = \int_{\mathbb{R}} f(t) U_t x \, dt. \quad (22)$$

The vector-valued integrand is continuous and compactly supported, and $\|T_f\| \leq \|f\|_{L^1}$. No operator-norm continuity of $t \mapsto U_t$ is needed for this definition.

Lemma 6.4. *The map $f \mapsto T_f$ is complex-linear and satisfies*

$$T_{f*g} = T_f T_g, \quad T_{f^*} = T_f^*. \quad (23)$$

Proof. The norm of the double-integral integrand applied to x is bounded by $|f(t)g(s)|\|x\|$, an integrable function. Vector-valued Fubini and the group law give the convolution identity. For the adjoint identity, pair (22) with a vector y , use unitarity to move U_t to U_{-t} , and change t to $-t$. The coefficient becomes $\overline{f(-t)}$, precisely the involution in theorem 2.1. $\square$

Boundedness alone is insufficient for taking a trace on an infinite-dimensional Hilbert space. For the next theorem assume $T_f \in \mathcal{S}_2(H)$, the Hilbert-Schmidt ideal, for every test f . Then $T_f T_g^*$ is trace class and

$$\langle T_f, T_g \rangle_{\mathcal{S}_2} = \text{Tr}(T_f T_g^*) \quad (24)$$

is a positive Hermitian form, linear in the first argument.

Theorem 6.5 (Cohomological trace transfer). *Suppose a geometric construction gives the positive pairing and flow above, the Hilbert-Schmidt condition for all T_f , and the comparison*

$$B_W(f, g) = \text{Tr}(T_f T_g^*) \quad (f, g \in \mathcal{T}). \quad (25)$$

Then $Q_W(f) \geq 0$ for every $f \in \mathcal{T}$.

Proof. Equation (25) with $g = f$ gives $Q_W(f) = \|T_f\|_{\mathcal{S}_2}^2$. $\square$

The theorem goes directly from geometric positivity and exact comparison to the classical quadratic form. It does not pass through a statement about zeros on a line. Its content as a research program is concentrated in the construction of the pairing and in (25), including the trace-ideal hypothesis. Neither is assumed to have been completed for the zeta function.

7 Cyclic and condensed constructions

7.1 Trace realization

Connes, Consani, and Marcolli construct a cohomological object using a cokernel in the category of cyclic modules associated with the inclusion of the idele class group into the adèle class space. Their cohomological Lefschetz formula identifies an arithmetic trace, while positivity is stated separately as an equivalent criterion [2, Theorem 6.1 and Proposition 6.2]. Their dictionary also identifies scaling correspondences and degree corrections; a suitable analogue of principal divisors and the subsequent effectivity argument remain part of the proposed geometry.

This work supplies an actual trace realization, rather than only names for cohomological propositions. It therefore provides a meaningful target for additional comparison maps. The task is to produce a positive structure compatible with that realization, or an intersection model whose signed pairing agrees with it. The trace identity itself cannot be used as the positive structure: a linear functional can agree with the explicit formula and still require proof of nonnegativity on convolution squares.

A useful comparison diagram would have the shape

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Gamma} & P \\ f \mapsto T_f \downarrow & & \downarrow J \\ \mathcal{S}_2(H) & \xlongequal{\quad} & \mathcal{S}_2(H), \end{array}$$

where J satisfies $-I(v, w) = \langle Jv, Jw \rangle_{\mathcal{S}_2}$ on the relevant primitive classes, and the trace realization satisfies (25). This diagram is a specification. It is not supplied by the logical assertion that an intersection condition and a spectral condition are each equivalent to the same proposition.

7.2 Restricted positive traces

The archimedean construction of Connes and Consani uses compression by the Sonin-space projection to obtain a positive ordinary trace functional, then compares it with an archimedean Weil distribution through a controlled correction. Their stated inequalities have support and transform-vanishing hypotheses [1, Theorems 1 and 11]. Such a result is precisely the kind of partial positive structure useful to the APP: it includes an actual Hilbert space, projection, and quantitative comparison. It does not establish unrestricted positivity for every finite set of places.

There is a second preservation issue here. If S is a projection and ϑ is a unitary representation, the compressed function $f \mapsto S\vartheta(f)S$ need not be multiplicative unless the range of S is invariant. Positivity of an expression such as $\text{Tr}(\vartheta(f)S\vartheta(f)^*)$ does not require that multiplicativity, but a representation-theoretic argument might. The needed property should be proved for the particular use rather than inferred from the word compression.

Connes's 2026 account explicitly leaves a simple, even ground-state condition and a sufficiently accurate comparison between a proxy kernel and the actual minimizing eigenfunction as remaining steps [9, Section 6.6]. Its geometric discussion does not remove these gaps. Suzuki's 2026 screw-function representation supplies another analytic comparison relevant to Part II [10]. A continuous kernel is not positive semidefinite merely by being a kernel for the correct quadratic form.

7.3 Coefficient categories

Condensed mathematics provides a setting for topological algebra with good categorical properties [11]. Genuine six-functor results then require specific geometric bases and coefficient categories. Mann's formalism concerns derived complete topological coefficients and p -torsion sheaves on suitable rigid-analytic objects and small v -stacks [12]. Anschütz, Le Bras, and Mann develop a related $\mathbb{Z}_p$ -linear formalism for solid quasi-coherent sheaves on the Fargues-Fontaine curve [13].

These results address construction and functoriality of categories, duality, and descent. To connect them to section 6, one still needs an arithmetic object, a realization in complex or real topological vector spaces, a cup/trace pairing with the required order property, and a comparison with W . A coefficient-change functor does not change the arithmetic base into a finite field. A descent theorem does not manufacture a global Frobenius or a real one-parameter flow without descent data for that action.

The distinction between purity and polarization is especially important. Purity is a statement about weights of an action under specified realizations. Polarization is additional pairing data and a positivity theorem with compatibility laws. Even in finite-dimensional linear algebra, an eigenvalue condition alone does not give those laws for a prescribed pairing. In an infinite-dimensional arithmetic setting, topology and domains add further requirements.

8 Preservation obstructions

8.1 Positivity certificates

Let B be any Hermitian form on a complex vector space V . A Hilbert-space factorization $B(v, w) = \langle jv, jw \rangle$ immediately proves $B \geq 0$. Conversely, if positivity is already known, the standard quotient construction produces a pre-Hilbert realization. This converse is useful for classification, but prevents the factorization alone from being advertised as an independent proof strategy.

Lemma 8.1. *Suppose $B(v, v) \geq 0$ for all v . Then*

$$|B(v, w)|^2 \leq B(v, v)B(w, w).$$

The set $N = \{v : B(v, v) = 0\}$ equals $\text{rad } B$, is a vector subspace, and the induced form on V/N is positive definite.

Proof. If $B(w, w) > 0$, expand $B(v + \lambda w, v + \lambda w)$ and choose $\lambda = -B(v, w)/B(w, w)$. Nonnegativity gives the inequality. If $B(w, w) = 0$ and $B(v, w) \neq 0$, choose λ with $\lambda B(v, w)$ real and arbitrarily negative; the same expansion contradicts nonnegativity. Thus $B(v, w) = 0$ whenever w is null.

Hermitian symmetry gives the corresponding statement in the other variable. Hence N is the radical, which is a subspace, and the quotient form is well-defined and positive definite. $\square$

The completion of this quotient is a Hilbert space. Applied to B_W , the construction starts with the global positivity assumption. Applied to a geometric intersection form whose negativity has been proved independently, it can instead be a valid later step in a geometric realization. The order of construction matters: taking the same algebraic quotient before knowing positivity does not turn an indefinite form into a Hilbert norm.

Part III gives a categorical account of these certificates and their generated realizations. The present conclusion is narrower: existence of a norm-square representation is equivalent certificate information at the level of Hermitian forms, whereas a proposed arithmetic geometry must be constructed by additional mathematics. No logical independence of the construction problem from RH is asserted.

8.2 Weights and Jordan blocks

Consider

$$A = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix} \quad \text{on } \mathbb{C}^2.$$

Every eigenvalue is imaginary, but

$$e^{tA} = e^{it} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The vector $(0, 1)$ has an orbit with unbounded norm as $|t| \rightarrow \infty$. Any two norms in finite dimension are equivalent, so no positive definite Hermitian norm can make this entire group isometric. Thus a centered eigenvalue condition does not imply unitarity in the presence of a Jordan block.

If a finite-dimensional operator is diagonalizable and its eigenvalues are imaginary, one can choose an eigenbasis and declare it orthonormal. That constructs some positive pairing. It does not show that a specified cup/trace pairing is positive or that the resulting choice is natural under arithmetic morphisms. Semisimplicity and a choice of basis solve a different linear-algebra problem from geometric polarization.

The infinite-dimensional distinction is stronger. A list of eigenvalues may fail to describe continuous spectrum, may lack a complete system of eigenvectors, or may concern vectors outside a proposed operator domain. The eigenvector consequence in theorem 6.3 is deliberately formulated so that none of those assertions is silently added.

8.3 Trace and subtraction

A supertrace does not preserve positivity of the underlying operator. On a graded space with one even and two odd one-dimensional components, the identity is positive in every component but has supertrace $1 - 2 = -1$. To use an alternating trace formula for a positivity argument, one must isolate the contribution of the relevant degree and identify the remaining terms.

A subtraction procedure also needs its own order theorem. Let P_n be the rank- n orthogonal projection on ℓ^2 . Each P_n is positive and $\text{Tr}(P_n) = n$. The renormalized expression $\text{Tr}(P_n) - (n + 1)$ is constantly -1 . This elementary example does not criticize any particular regularization; it proves that positivity of the cutoff operators alone does not justify positivity after subtracting a counterterm.

Accordingly, a regularized determinant or a distributional trace may be part of an arithmetic comparison without being the positive trace in theorem 6.5. To replace the ordinary trace there,

one must prove that the chosen functional is nonnegative on the relevant operator squares. If this condition is asserted only after identifying those squares with all tests in B_W , it is again the target in another notation.

8.4 Mixed pairings and gluing

Let B on $\mathbb{R}^2$ have matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The restriction to each coordinate line is positive definite, but $B((1, -1), (1, -1)) = -2$. Positivity on local pieces is therefore insufficient without information about mixed pairings. In a descent construction, compatibility must control the form on overlaps and on combinations of locally represented vectors, not merely the diagonal form on each isolated piece.

There is a useful case where no further limit is needed. If V is the increasing union of subspaces V_a , and the restriction of a single form B to each V_a is nonnegative, then B is nonnegative on V . Every vector is contained in one stage. But the phrase “a single form” is doing work: unrelated forms B_a on unrelated spaces do not meet that hypothesis. Comparison maps must identify them with restrictions of the target, or convergence must be proved in an appropriate sense.

This is why support-local arithmetic positivity is a promising specification. The spaces $\mathcal{T}_a$ really do exhaust $\mathcal{T}$. The prime-power terms are exactly finite at each stage. The remaining task is to construct compatible positive representations of the whole signed form at those stages. Positivity of separate local factors is not a substitute.

9 The formal context

9.1 Statements and geometric objects

The five associated research repositories contain both references to established mathematics and Lean declarations with deliberately schematic semantics. The distinction is necessary when they are used as inputs to a new positivity program. The following observations concern the cited source declarations; they do not treat a paper citation as a Lean import or infer a proof from a theorem’s name [14].

In `hott-foundations-mathematics`, the file `lean/langlands-analytic-number-theory/Zeta.lean` imports Mathlib’s actual Riemann zeta function and defines an RH-style proposition at lines 32-33. Its translation-status structure at lines 45-52 records several analytic components as unfinished. This gives a concrete classical statement. It does not supply a geometric realization of the Weil form.

In `hott-riemann-hypothesis`, `lean/oq6-rh-as-hott-proposition/RHProp.lean`, lines 28-39, defines an RH-style proposition and proves proof irrelevance and excluded middle for it. Neither theorem establishes a chosen disjunct or produces a geometric pairing. The two certifier declarations at lines 53 and 64 contain `sorry`. None of these declarations can serve as an independently proved positivity input.

Both early RH-style definitions omit the pole exclusion $s \neq 1$ present in the pinned Mathlib definition `RiemannHypothesis`. Their equivalence with that standard proposition therefore needs a comparison concerning the totalized pole value. We make no assertion that the assigned value `riemannZeta(1)` is zero. This distinction is separate from the geometric realization problem.

In Volume IV, `lean/Oq5LanglandsTopos.lean`, lines 45-56, represents the purported topos and its morphisms by `Unit`. Four structural predicates at lines 66-80 are defined as `True`. A singleton

model may be adequate for checking the syntax of an interface. It is not the construction of a category of condensed arithmetic spaces with the objects and pairings required here.

9.2 The two Volume V bridges

Volume V's Track A formulation uses `ECond := Unit`, a self-adjointness predicate defined as `True`, and a universal spectrum in `V5aTrackAFormulation/TrackAFormulation.lean`, lines 54 and 83-91. Its axiom `Infra1_HP_collapse`, lines 186-193, directly supplies the real-part conclusion about actual zeta zeros from the schematic spectral data. The reverse bridge at line 226 is an axiom as well. These declarations do not construct a self-adjoint arithmetic operator.

The direct Track A obstruction file belongs to a separate abstract predicate setup. It does not import the full geometric infrastructure as a transitive mathematical realization. A theorem that packages an abstract obstruction is valid conditional logic, but its existence does not establish the intersection or trace comparisons in this paper.

Track B exhibits a more immediate semantic gap. In `Infra5CondensedEtaleCohomology/SixFunctor.lean`, lines 27-42, stacks, morphisms, and sheaves are `Unit`. In `LefschetzTrace.lean`, lines 31-37 and 48-68, dualizability is `True`, both traces are zero, and the displayed trace theorem reduces to $0 = 0$.

The same distinction persists in `DelignePurity.lean`. Its weight-zero predicate at line 31 is `True`; its purported eigenvalue relation at lines 40-42 ignores integer degree and eigenvalue arguments; and its purity-bound predicate at line 75 is `True`. The theorem at lines 86-94 consequently proves `True`, not an equality $|\alpha| = q^{i/2}$. The real Deligne theorem cannot be recovered from that conclusion by interpreting the comments as extra hypotheses.

The cross-track equivalence is also explicit about its logical route. In `V5bTrackBFormulationAndEquivalence/TrackBFormulationAndEquivalence.lean`, `cohomRH_forward` and `cohomRH_reverse` are axioms at lines 179 and 185. The theorem `RH_TrackA_iff_TrackB`, lines 211-220, composes those hooks with the Track A equivalence through classical RH. The separate `HadamardLefschetzAgree` proposition and its inhabitant are axioms at lines 228-231.

Thus the supplied code contains no direct pairing-preserving map from a cohomological object to the spectral object. The APP asks for such a map with concrete source and target. Replacing one equivalence axiom by another internal equivalence axiom would not answer that question.

9.3 Carrier transport in Volume VI

Volume VI's `Vol5_5/NoPhantomAssembly.lean`, lines 23-56, assembles a no-phantom conclusion from a structure containing detector, rational-dilation, orthogonality, and admissibility inputs. The assembly does not construct those inputs. Their names do not establish that they hold for the intended arithmetic realization.

The file `Closure/OffCriticalDefectKernelBridge.lean`, lines 59-99, gives a particularly useful diagnostic. Its unrestricted transport object is equivalent to nonemptiness of a canonical model carrier: an element of the target defines constant transport, and a fixed nonempty source packet extracts an element from any transport. The proof is valid for the displayed type of maps.

That type does not require linearity, preservation of a pairing, compatibility with arithmetic operators, or transport of a genuinely nonzero vector. A constant map cannot in general satisfy those stronger conditions. Accordingly, this carrier bridge cannot be used as the comparison map in theorem 3.1 or theorem 6.5. The defect is one of formulation: the map type forgets the properties needed by the intended application.

10 Construction requirements

10.1 The intersection route

A concrete intersection proposal should begin by choosing its arithmetic category and its object. For an ordinary arithmetic surface, this means specifying the surface and metrics. For a noncommutative or condensed proposal, it means specifying the replacement for divisor classes, intersection, and the primitive condition. The adjective arithmetic supplies none of these operations on its own.

The next step is a raw correspondence assignment $C(f)$ on a stated test class. Before attempting positivity, one can investigate linearity, convolution compatibility, and the adjoint relation. These are independently meaningful construction problems. If degree and codegree functionals exist, theorem 3.3 then provides an explicit normalization target. The primitive condition can be checked separately from the sign of the pairing.

The remaining arithmetic comparison has a definite shape:

$$B_W(f, g) = -I(Cf, Cg) + a(Cf)\overline{b(Cg)} + b(Cf)\overline{a(Cg)}. \quad (26)$$

This formula uses the Hermitian extension of the real degree-normalization calculation. It must be derived from the arithmetic construction, not used as the definition of a positive form whose existence already presupposes the desired sign. It can first be tested on a class for which every local term is explicitly calculable.

If the comparison is initially available only on a subspace $D \subset \mathcal{T}$, two separate extension statements are needed: density of D in the chosen topology and continuity of both sides of the comparison. Moment restrictions such as $\mathcal{T}_0$ do not meet that density condition in the test topology. A different extension theorem may exist, but its proof is part of the program.

An index inequality on the full primitive space is a possible known input when the candidate geometry falls under an established arithmetic Hodge theorem. If the candidate lies outside that setting, the index inequality is a new geometric theorem to be proved. Its status should be determined after the category is fixed. There is no useful global answer to whether “arithmetic positivity” is known without specifying the arithmetic object and its pairing.

10.2 The cohomological route

For a cohomological candidate, the data of (16) should be constructed before claiming a positive Hilbert realization. This includes a real or complex realization functor, an actual trace in the required degree, and an involution or Hodge operator. The positivity theorem must concern this pairing and its arithmetic origin, rather than an arbitrarily chosen inner product on a vector space with the desired eigenvalues.

The compatibility equations (17)-(19) then reduce the centering calculation to theorem 6.1. Strong continuity provides the analytic hypothesis for theorem 6.2. If only an infinitesimal operator is constructed, its domain and the conditions for integrating it must be supplied separately. An integration theorem cannot be replaced by the eigenvector calculation.

The final comparison is the ordinary trace identity (25). To establish it, one needs the trace ideal, the identification of the middle contribution, and control of the terms removed from a cohomological alternating trace. A determinant formula may help prove this equality through an explicit-formula theorem, but it does not bypass the trace-class or positivity requirements.

These tasks can proceed in parallel because several are meaningful before the complete arithmetic object is known. One may prove a functoriality law, compute a local trace, or establish a norm

Table 1: Each arrow in a direct positivity argument has a separate preservation obligation. The table describes required mathematical data, not completed arithmetic constructions.

Map	Source and target	Required equation or property
Test realization	Tests to primitive classes	The Weil pairing equals the negative intersection pairing.
Primitive projection	Raw classes to normalized classes	Degree and codegree vanish with the explicit correction formula.
Cohomological realization	Cohomology to a pre-Hilbert space	Cup product and trace define the proved positive pairing.
Centering	Conformal flow to isometric flow	The weight-one factor cancels under multiplication by $e^{-t/2}$.
Integrated action	Tests to bounded operators	Convolution and involution become composition and adjoint.
Trace comparison	Operator squares to scalars	The ordinary trace is defined and equals the complete Weil form.
Analytic extension	A specified dense class to all tests	Both sides are continuous in the same topology.

estimate in a candidate category. What is not legitimate is to combine separately named properties without checking that they concern the same realization and the same maps.

10.3 A preservation ledger

A logical equivalence between the propositions “the source is positive” and “the target is positive” is absent from this table. Such an equivalence can record the strength of a conjecture but cannot supply any of the displayed maps. The ledger instead asks which operations exist and which identities they preserve.

The same requirement applies to comparisons between different geometric proposals. An isomorphism of bare vector spaces is insufficient. A functor between coefficient categories must carry the relevant objects, actions, pairings, and trace maps to the specified targets. If it preserves only some of them, the remaining preservation equations are explicit residual problems. This is a more useful account of a bridge than an unqualified claim that two frameworks encode the same arithmetic.

11 Formal boundaries

The companion Lean development is organized around unconditional consequences of supplied data. The real vector-space formulation applies to the underlying real space of the full complex test algebra; it does not restrict the admissible functions. Its parameters are arbitrary forms and maps, so the transfer proofs cannot silently import RH through a concrete arithmetic instance.

In `APP/Structures.lean`, `FormComparison` records equality of a source bilinear form with the pullback of a target form. Its diagonal and nonnegativity consequences use that equality. `PrimitiveComparison` records primitive membership and a negative diagonal comparison. Its nonnegativity theorem takes an index inequality as an explicit hypothesis. A nonzero example uses the indefinite plane from section 3.

In `APP/Bridges.lean`, the namespace `PrimitiveProjection` proves orthogonality of the normalized class, the corrected intersection identity, and positivity under an exact comparison. These are the algebraic content of theorem 3.3. The mixed-pairing counterexample is checked

as `Countermodels.positive_axes_do_not_glue`. The module `APP/Spectral.lean` treats the weight identity on a specified domain and the nonzero-eigenvector consequence. It supplies neither a cohomological object nor a correspondence between its vectors and classical zeta zeros.

The analytic statements in section 2, the cohomological completion and integrated-action discussion, and the primary-source geometry are mathematical arguments and cited results in this paper. They are not asserted to be a completed Lean formalization of the classical Weil distribution. In particular, the project does not declare an axiom that identifies an abstract positive form with B_W . An arithmetic instance of that identification is the central remaining construction.

This boundary is useful rather than cosmetic. A small checked theorem can certify that a proposed comparison, if constructed, suffices. It cannot certify that the comparison exists. Conversely, a substantial arithmetic construction may be valuable before its full analytic formalization is complete, provided the theorem and its proof are stated accurately and the formal coverage is not exaggerated.

12 The remaining arithmetic maps

The available mathematics supports several concrete next steps. Arithmetic Hodge index can be used once a suitable space of classes and primitive realization are supplied. A positive cohomological pairing can be centered to produce isometries once its flow compatibilities are proved. A genuine cyclic trace realization can be compared with either of these structures. Restricted positive trace estimates can guide a support-local construction, provided their error and extension conditions are retained.

The unresolved objects are correspondingly specific. For the intersection route they are the arithmetic correspondence assignment, its degree normalization, and the signed intersection comparison with the full Weil form. For the cohomological route they are the arithmetic positive pairing, the compatible action, and the ordinary trace-ideal comparison. Condensed or noncommutative categories may provide a suitable home for these constructions, but their general formalism does not establish the arithmetic equations.

These specifications are smaller units of work within a research program. We make no claim that their conjunction is logically weaker than global positivity, that any individual arithmetic construction is independent of RH, or that a universally minimal geometry has been identified. The point of the decomposition is that each proposed map can be constructed, falsified, or refined using its own preservation equations. A successful program must eventually prove those equations for an independently defined arithmetic object.

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